POSTGRADUATE THIRD SEMESTER EXAMINATIONS, 2021

Subject: MATHEMATICS **Course ID: 32154**

Course Code: Math-304ME (Old) **Course Title: Advanced Algebra-I (Old)**

Full Marks:40 **Time: 2HRS**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning.

Answer five(5)from the following eight(8) questions.$\mathbf{8 \times 5 = 40}$

1. **a)** Let R be an integral domain and M be an R-module. Prove that $Tor(M)$, the set of all torsion elements of M, is a submodule of M. Give an example of a ring R and an R-module M such that Tor(M) is not a submodule.$\mathbf{2 + 2 = 4}$

   b) Let I be a right ideal of a ring R and let N be its annihilator in M. Prove that the annihilator of N in R contains I.

   Give an example where the annihilator of N in R does not equal I.$\mathbf{1 + 3 = 4}$

2. **a)** Find $Hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z})$. $\mathbf{3}$

   b)Show that a submodule of a cyclic module is not always cyclic. $\mathbf{2}$

   c) Define linear independence of a subset in a module.

   Show that amaximallinearly independent set neednot be a basis in a module.$\mathbf{1 + 2 = 3}$

3. **a)**Let $N_1, N_2, \dots, N_k$be submodules of an R-module M. Then show that the following are equivalent:

   (i) $N_1 \times N_2 \times \dots \times N_k \cong N_1 + N_2 + \dots + N_k$;

   (ii) $N_j \cap (N_1 + \dots + N_{j-1} + N_{j+1} + \dots + N_k) = 0$ for all $j = 1,2, \dots, k$;

   (iii) Every element of $N_1 + N_2 + \dots + N_k$ can be uniquely written. $\mathbf{5}$

   b)Show that an R-module M is irreducible if and only if $M \neq 0$ and M is a cyclic module.$\mathbf{3}$

4. **a)** Define an Artinian R-module. Let M be a Noetherian R-module. Prove that every non-empty set of submodulesof M contains a maximal element. $\mathbf{1 + 3 = 4}$

b) Define free module.Show that submodule of a free module need not be free always.$\mathbf{1 + 3 = 4}$

5. **a)** Definelocal ring. Prove that a commutative ring R with 1 is a local ring if and only if there exists a maximal ideal M of R such that every element of the form $1 + m$ with $m \in M$,is a unit in R.$\mathbf{1 + 4 = 5}$

b) Let $A_1, A_2, \dots, A_n$be ideals of a commutative ring R with 1. If each A_i, A_j $(i \neq j)$ are co-maximal, then show that there is a surjection from $R$to $\frac{R}{A_1} \times \ldots \times \frac{R}{A_n}$.**3**

6. **a)**For any two ideals I and J, prove that $rad(I + J) = rad(rad\ I\ +\ rad\ J)$. **3**

b)Is contraction of a maximal ideal always maximal? Justify. **1**

c)Suppose there is a homomorphism from a ring R to a ring R_1. Prove that the set of all contracted ideals of R and the set of all extended ideal of R_1 are in a bijective correspondence. **4**

7.a) Let P be a prime ideal of a ring R. Define localization of R at P. Find one of its maximal ideal. $\mathbf{1 + 2 = 3}$

b)Show that $S^{-1}(rad\ I) = rad(S^{-1}I)$, for any ideal I of R. **3**

c) Suppose there is a homomorphism from a ring R to a ring R_1 and P is a prime ideal of R. If $P^{ec} = P$, then show that P is the contraction of a prime ideal of R_1. **2**

8.a) Define integral extension. Show that the integrality property is transitive. $\mathbf{1 + 3 = 4}$

b)Let S be integral over R. If T is a multiplicatively closed subset of R, then prove that $T^{-1}S$ is integral over $T^{-1}R$. **2**

c) Prove Nakayama's lemma.**2**

END